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EFFECTIVE NUMERICAL–ANALYTICAL SOLUTION OF ISOPERIMETRIC VARIATIONAL PROBLEMS OF MECHANICS BY AN ACCELERATED CONVERGENCE METHOD†

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A new numerical–analytical method for solving non-linear variational problems of mechanics is presented. The method enables additional isoperimetric conditions and boundary conditions of different types to be taken into account. Unlike existing approaches, the method is based on the use of residuals of the unknown functions corresponding to abscissae at which they satisfy the required conditions. The method has a clear geometrical interpretation and provides a more confident idea as to the convergence of the iteration algorithm, which is quadratic in nature. The algorithm constructed is used to compute single-mode and multi-mode viscous incompressible flows in a plane convergent channel (Jeffrey–Hamel flow) for a broad range of governing parameters (the aperture angle and Reynolds number).

1. FORMULATION OF THE PROBLEM

The basic principle of the proposed numerical–analytical method for solving isoperimetric problems may be demonstrated by taking, as an example, the simplest classical problem of the form [1, 2]

$$J[y]=\int_0^l F(x,y,y')dx \to \min_y, \quad I[y]=\int_0^l N(x,y,y')dx=0 \tag{1.1}$$

$$y=y(x), \quad y(0)=y(l)=0, \quad 0\le x\le l, \quad (y,y')\in G$$

Certain more general specifications of the boundary values of x, y and the quantity I can be reduced to this case by a simple transformation. It is assumed that the functions F and N are fairly continuous and are such that the necessary conditions for optimality, in the form of a boundary-value problem for the Euler–Lagrange equations, are applicable [1, 2]. As a result one arrives at the following relations that determine the unknown continuous function $y(x)$ and an unknown Lagrange multiplier λ:

$$L[K]=0, \quad y=y(x,\lambda), \quad y(0,\lambda)=y(l,\lambda)=0$$

$$z'=N, \quad z=z(x,\lambda), \quad z(0,\lambda)=z(l,\lambda)=0 \tag{1.2}$$

$$K=K(x,y,y',\lambda)\equiv F+\lambda N, \quad L[\cdot]\equiv \frac{d}{dx}\frac{\partial}{\partial y'}(\cdot)-\frac{\partial}{\partial y}(\cdot)$$

Thus, it is required to solve non-linear boundary-value problem (1.2) for y and z and to select the optimal solution from among the solutions found in accordance with conditions (1.1). As far as computations are concerned, one has to determine the unknown parameters $\gamma = y'(0)$ and λ, i.e. the data still needed to integrate the Cauchy problem corresponding to (1.2).

None of the effective methods available for investigating non-linear boundary-value problems of type (1.2) is convenient for investigating essentially non-linear mechanical systems over a wide range of governing parameters. The numerous available methods – the shooting, double-sweep, gradient descent, successive approximation or tangents, quasi-linearization and the many modifications of these methods [1–7] – involve considerable difficulties, of which the following deserve mention: divergence, or extraordinarily slow, doubtful, convergence, of iterations (degeneracy of the sensitivity matrix and "ravine" effects), very long processor time and inability in practice to check the convergence and actual precision

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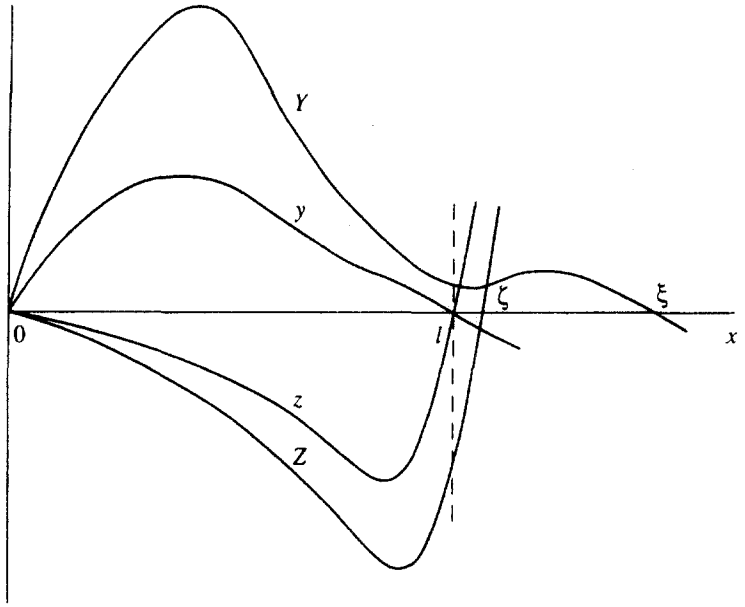

Fig. 1

of the approximate solutions. This situation may apparently be attributed primarily to disregard of an important property of boundary-value problems: the solution depends in an essential way on the length of the interval l. In all approaches that have been proposed, the interval over which the argument x varies is fixed and is not varied during the investigation of the approximate solutions. Applicable procedures use only the residuals of the unknown functions at fixed points (for problem (1.2) – at $x = l$ and $x = 0$). There is no investigation of the behaviour of the approximate solutions based on integrating the corresponding Cauchy problems and determining their residuals corresponding to x; neither are such measures used to improve the accuracy of the missing data (in particular, γ and λ).

The desirability of using one of the basic properties of the approximate solution of problem (1.2) – the residual corresponding to the argument x – may be illustrated graphically (see Fig. 1). Let Y, Z be a solution of the Cauchy problem corresponding to Eqs (1.2) and satisfying the conditions $Y = 0$, $Y' = \gamma^*, Z = 0, \lambda = \lambda$ at $x = 0$ (where γ^* and λ^* are certain estimates of the parameters γ and λ). The functions y and z at $x = 0$ satisfy the conditions $y(0) = z(0) = 0; y'(0) = \gamma^0$, and $\lambda = \lambda^0$, where γ^0 and λ^0 are the unknown exact values. Investigation of the residual of Y for $x \geqslant l$ indicates that Y is only slightly sensitive to the parameter γ. Although the residual corresponding to Y is numerically small, the number γ^* cannot be considered a good approximation, since the magnitude of the root of the function $Y(x, \gamma^*, \lambda^*)$ closest to l (denoted, say, by ξ) is significantly different from l: $|\xi - l| \sim l$ (the residual corresponding to x is large). Conversely, the function Z may depend very much on the parameter λ, that is, a small change in λ may cause significant changes in the residual $Z(l, \gamma^*, \lambda^*)$. For example, this residual may be relatively large, while the residual corresponding to the abscissa x will be small (the value ζ of the root of the function Z closest to l, that is, $|\zeta - l| \ll l$). In that sense, λ^* may be regarded as a "good" approximation to λ.

Thus, an analysis of the behaviour of the solution $Y(x, \gamma^*, \lambda^*), Z(x, \gamma^*, \lambda^*)$ yields additional information about its sensitivity to changes in the unknown parameters γ and λ. Without such an analysis, it is difficult to judge whether a solution of the boundary-value problem exists in the neighbourhood of estimates γ^* and λ^* found in some way. The most natural and widely used methods for constructing estimates γ^* and λ^* are the variational-functional approaches based on the Rayleigh–Ritz and Bubnov–Galerkin methods [3–4].

An investigation of the behaviour of residuals and the construction of numerical solutions of multidimensional boundary-value problems encounter considerable difficulties [3–7]. This is the case even for one-dimensional non-linear systems, in particular, for discontinuous systems. The main difficulties arise in the irregular case when the equations for parameters like γ and λ, which determine the boundary conditions, have a large number of roots, e.g., in the case of oscillatory systems [8–10].

Figure 2 demonstrates a different type of behaviour of the residuals $Y(l, \gamma, \lambda^*)Z(l, \gamma^*, \lambda)$ as functions of γ and λ, respectively. The initial zone 1 corresponds to the regular situation (the required solution

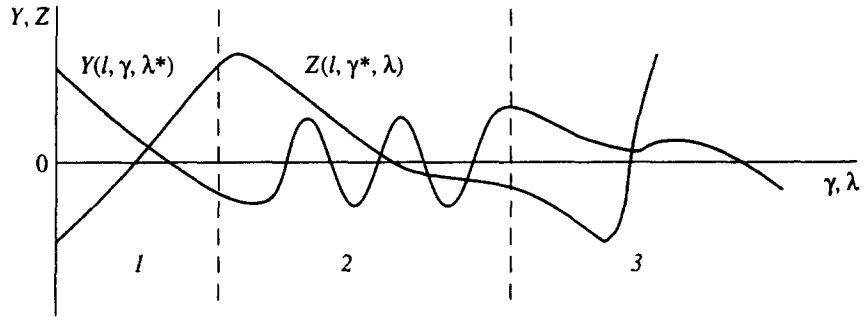

Fig. 2

exists in a small neighbourhood of γ^*, λ^*). In zone 2 one observes a weak dependence of Z on λ and rapid oscillations of Y as a function of λ, while in zone 3 there are relatively sharp changes in the residual corresponding to Z but weak changes in that corresponding to Y; sections 2 and 3 are not regular in the computational sense. In many cases, further simplification is necessary, such as the use of asymptotic small-parameter methods (singular perturbations [11–13] or averaging [10]).

In what follows we propose a numerical-analytical accelerated-convergence method for solving boundary-value problems of type (1.2), based on allowance for the residuals of the approximate solutions corresponding to the argument x. Computational experience indicates that this approach yields high precision and stability of the computations, so that one can effectively control the convergence of the approximate numerical calculations and construct the required solutions in complex irregular situations, while allowing the governing parameters of the system to vary over a wide range.

2. A STANDARD PROCEDURE, OF THE SAME TYPE AS THE METHOD OF TANGENTS AND ITS MODIFICATION

We shall describe, as it applies to problem (1.2), a Newton-type method for the numerical determination of the unknown parameters γ and λ, defining the corresponding Cauchy problem. We introduce sensitivity functions v, w and u, s, described by the linear relations

$$L[K'_y v + K'_{y'} v'] = 0, \quad w' = N'_y v + N'_{y'} v'$$

$$v(0) = 0, \quad v'(0) = 1, \quad w(0) = 0$$

$$L[K'_y u + K'_{y'} u' + N] = 0, \quad s' = N'_y u + N'_{y'} u' \tag{2.1}$$

$$u(0) = u'(0) = s(0) = 0$$

$$v = \frac{\partial y}{\partial \gamma}, \quad w = \frac{\partial z}{\partial \gamma}; \quad u = \frac{\partial y}{\partial \lambda}, \quad s = \frac{\partial z}{\partial \lambda}$$

A prime with a subscript y or y' denotes partial differentiation with respect to y or y', respectively. If the function $y(x)$ is known, a solution of Cauchy problem (2.1) is constructed numerically, integration being performed independently for v, w and u, s. The required function $y(x)$ is defined by the relations

$$L[K] = 0, \quad y(0) = 0, \quad y'(0) = \gamma; \quad y = y(x, \gamma, \lambda), \quad y(l, \gamma, \lambda) = 0$$

$$z' = N, \quad z(0) = 0; \quad z = z(x, \gamma, \lambda), \quad z(l, \gamma, \lambda) = 0 \tag{2.2}$$

Suppose an initial approximation (estimate) γ^* and λ^* is known for the parameters γ and λ and a irregular situation occurs (see Section 1 and Fig. 2 [zone 1]). Then one can use a Newton-type recurrence procedure to obtain more accurate values of γ and λ, based on high-precision integration of Cauchy problems (2.1) and (2.2).

Let us consider the first step of the procedure. According to (2.2), taking relations (2.1) into consideration, we have the following approximate expressions for y and z

$$y(x,\gamma,\lambda) \equiv y(x,\gamma^* + \delta\gamma, \lambda^* + \delta\lambda) = Y^*(x) + V^*(x)\delta\gamma + U^*(x)\delta\lambda + O_2^y(\delta\gamma,\delta\lambda)$$

$$z(x,\gamma,\lambda) \equiv z(x,\gamma^* + \delta\gamma, \lambda^* + \delta\lambda) = Z^*(x) + W^*(x)\delta\gamma + S^*(x)\delta\lambda + O_2^z(\delta\gamma,\delta\lambda) \tag{2.3}$$

$$\delta\gamma \equiv \gamma - \gamma^*, \quad \delta\lambda \equiv \lambda - \lambda^*$$

$$Y^*, Z^*, V^*, W^*, U^*, S^* \equiv y, z, v, w, u, s\,|_{\gamma=\gamma^*,\lambda=\lambda^*}$$

where Y^*, Z^* is a solution of Cauchy problem (2.2) for the known γ^*, λ^*; V^* and W^*, and U^*, S^* is the corresponding solution of (2.1), and $O_2^{y,z}$ is a term of the order of $(\delta\gamma^2, \delta\lambda^2)$. These functions are assumed to be known apart from $O(\delta\gamma^2 + \delta\lambda^2)$. Boundary conditions of (2.2) yield approximate equations for $\delta\gamma$ and $\delta\lambda$

$$V^*(l)\delta\gamma + U^*(l)\delta\lambda = -Y^*(l) + O_2^y$$

$$W^*(l)\delta\gamma + S^*(l)\delta\lambda = -Z^*(l) + O_2^z \tag{2.4}$$

In the general situation, the "small" quantities $Y^*(l)$, $Z^*(l)$ and $\delta\gamma$, $\delta\lambda$ may be of different orders. For the regular case, omitting quadratic terms, we find the unknown $\delta\gamma$ and $\delta\lambda$ from system (2.4) with an error of the same order O_2.

$$\delta\gamma^* = -\frac{Y^*(l)S^*(l) - Z^*(l)U^*(l)}{\Delta^*(l)}, \quad \delta\lambda^* = \frac{Y^*(l)W^*(l) - Z^*(l)V^*(l)}{\Delta^*(l)} \tag{2.5}$$

$$\Delta^*(l) = V^*(l)S^*(l) - U^*(l)W^*(l) \neq 0$$

The eigenvalues of the matrix of the coefficients of $\delta\gamma^*$ and $\delta\lambda^*$ must be quantities of the same order. Using the explicitly found expressions (2.5) for the corrections $\delta\gamma^*$ and $\delta\lambda^*$, we obtain improved values for the unknowns γ and λ

$$\gamma^{(1)} = \gamma^{(0)} + \delta\gamma^{(0)}, \quad \lambda^{(1)} = \lambda^{(0)} + \delta\lambda^{(0)}, \quad \gamma^{(0)} = \gamma^*, \quad \lambda^{(0)} = \lambda^*$$
$$|\gamma - \gamma^{(1)}| = O_2^y, \quad |\lambda - \lambda^{(1)}| = O_2^z \tag{2.6}$$

The first step of the iterations has thus been completed. On that basis, using relations of type (2.1)–(2.6), one now performs the next, second step, determining $\gamma^{(2)}$ and $\lambda^{(2)}$. The $(n+1)$st step of the recurrence procedure prescribed by the method of tangents reduces to simultaneous integration of the Cauchy problems (2.2) with known $\gamma = \gamma^{(n)}$ and $\lambda = \lambda^{(n)}$ and simultaneous integration of the Cauchy problems (2.1); as a result one obtains the unknown functions $Y^{(n)}(x)$ and $Z^{(n)}(x)$ and sensitivity functions $V^{(n)}(x)$, $W^{(n)}(x)$, $U^{(n)}(x)$ and $S^{(n)}(x)$. The use of an expression of type (2.3)

$$y(x,\gamma^{(n)},\lambda^{(n)}) = Y^{(n)}(x) + V^{(n)}(x)\delta\gamma^{(n)} + U^{(n)}(x)\delta\lambda^{(n)} + O_2^y(\delta\gamma^{(n)},\delta\lambda^{(n)}) \tag{2.7}$$

$$z(x,\gamma^{(n)},\lambda^{(n)}) = Z^{(n)}(x) + W^{(n)}(x)\delta\gamma^{(n)} + S^{(n)}(x)\delta\lambda^{(n)} + O_2^z(\delta\gamma^{(n)},\delta\lambda^{(n)})$$

with $x = l$ leads, according to (2.2), to equations of type (2.4) for $\delta\gamma^{(n)}$ and $\delta\lambda^{(n)}$, which are then solved with an error $O_2(\delta\gamma^{(n)}, \delta\lambda^{(n)})$ to determine the unknowns $\delta\gamma^{(n)}$ and $\delta\lambda^{(n)}$ and improved values $\gamma^{(n+1)}$ and $\lambda^{(n+1)}$

$$\delta\gamma^{(n)} = -\frac{Y^{(n)}(l)S^{(n)}(l) - Z^{(n)}(l)U^{(n)}(l)}{\Delta^{(n)}(l)}$$

$$\delta\lambda^{(n)} = \frac{Y^{(n)}(l)W^{(n)}(l) - Z^{(n)}(l)V^{(n)}(l)}{\Delta^{(n)}(l)}$$

$$\Delta^{(n)}(l) = V^{(n)}(l)S^{(n)}(l) - U^{(n)}(l)W^{(n)}(l) \neq 0 \tag{2.8}$$

$$\gamma^{(n+1)} = \gamma^{(n)} + \delta\gamma^{(n)}, \quad |\gamma - \gamma^{(n+1)}| \leqslant O_2^{y(n+1)}$$

$$\lambda^{(n+1)} = \lambda^{(n)} + \delta\lambda^{(n)}, \quad |\lambda - \lambda^{(n+1)}| \leqslant O_2^{z(n+1)}$$

Substituting the quantities (2.8) into (2.7) we obtain a solution in the $(n+1)$st approximation. As remarked, the convergence of the procedure is satisfactory in the regular case; in irregular situations the computation process may diverge, as often happens in procedures of continuation with respect to a parameter (see Section 5). The scheme presented above is equivalent to minimizing a quadratic residual, e.g. of the form

$$E^2(\gamma,\lambda)=Y^2(l,\gamma,\lambda)+Z^2(l,\gamma,\lambda)\to\min_{\gamma,\lambda},\quad \gamma\in\Gamma,\quad \lambda\in\Lambda \tag{2.9}$$

Lacking an analysis of the residual as a function of l, one cannot confidently evaluate the quality of the computation process and its convergence (see the example in Section 5). It should also be noted that the computation process and an estimate of the accuracy are considerably complicated by the multi-dimensional nature of the problem: one has to integrate a ninth-order system for $Y, Y', Z; V, V', W; U, U', S$. At the same time one has to solve a system of two equations for γ and λ, which are generally determined numerically with different degrees of precision. As an effective approach to the computation of a highly accurate value of γ^0 and λ^0, one should use a numerical-graphical method, based on constructing the functions $\gamma(\lambda)$ and $\lambda(\gamma)$ in some neighbourhood of the known point γ^*, λ^*, though these functions need not be single-valued (see below, Sections 3 and 4). Each curve is constructed separately by solving one-dimensional boundary-value problems using a rapidly convergent accelerated convergence method and the procedure of continuation with respect to a parameter. In a small neighbourhood of the point at which the curves intersect, one can use a high-order approximation. After improving the values of γ and λ, the computation algorithm continues in recurrence fashion. To obtain a confident idea of the existence of a solution to the boundary-value problem, one has to determine and use the residual corresponding to the abscissa x. In practical computations, such an approach turns out, in the final analysis, to be effective [14, 15].

The standard, Newton-type procedure (2.1)–(2.9), which is poorly convergent, may be modified to make use of the residuals of the solutions $Y^{(n)}(x), Z^{(n)}(x)$ corresponding to x. These residuals are defined by analogy with the scalar case [14]

$$\begin{aligned}
&\varepsilon^{(n)}=1-\frac{\xi^{(n)}}{l},\quad \xi^{(n)}=\arg\min_{\xi_i^{(n)}}\left|1-\xi_i^{(n)}\right|\\
&\xi_i^{(n)}=\mathrm{Arg}_x\, Y^{(n)}(x),\quad |\varepsilon^*|\ll 1\\
&\mu^{(n)}=1-\frac{\eta^{(n)}}{l},\quad \eta^{(n)}=\arg\min_{\eta_i^{(n)}}\left|1-\eta_i^{(n)}\right|\\
&\eta_i^{(n)}=\mathrm{Arg}_x\, Z(x),\quad |\mu^*|\ll 1
\end{aligned} \tag{2.10}$$

In the linear approximation with respect to $\varepsilon^{(n)}$ and $\mu^{(n)}$, the expressions for the corrections $\delta\gamma^{(n)}$ and $\delta\lambda^{(n)}$ may be written in the form

$$\begin{aligned}
&\delta\gamma^{(n)}=-\frac{\varepsilon^{(n)}Y'^{(n)}(l)S^{(n)}(l)-\mu^{(n)}Z'^{(n)}(l)U^{(n)}(l)}{\Delta^{(n)}(l)}\\
&\delta\lambda^{(n)}=\frac{\varepsilon^{(n)}Y'^{(n)}(l)W^{(n)}(l)-\mu^{(n)}Z'^{(n)}(l)V^{(n)}(l)}{\Delta^{(n)}(l)}\\
&\varepsilon^{(n+1)}\sim(\varepsilon^{(n)})^2,\quad \mu^{(n+1)}\sim(\mu^{(n)})^2
\end{aligned} \tag{2.11}$$

Implementation of operations (2.10) to determine the residuals $\varepsilon^{(n)}$ and $\mu^{(n)}$ yields a more confident idea as to whether a solution $y(x), z(x)$ near $Y^{(n)}(x), Z^{(n)}(x)$ exists, and the validity of the estimates (2.11) for $\varepsilon^{(n+1)}$ and $\mu^{(n+1)}$ provides an indication as to the actual rate of convergence of the process to the solution. Violation of these estimates indicates that the iterative procedure will diverge; in such cases, however, it may converge to another solution (bifurcation).

3. AN ACCELERATED CONVERGENCE METHOD BASED ON THE RESIDUAL CORRESPONDING TO THE ABSCISSA

Divergence of the standard procedure proposed in Section 2 for the simultaneous determination of the unknown parameters γ and λ may be avoided by adopting a numerical-analytical approach based

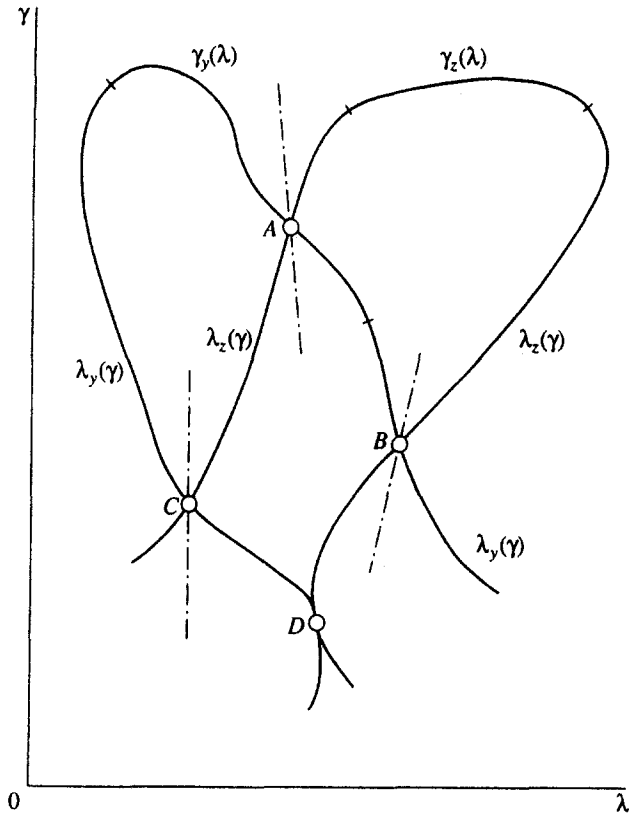

Fig. 3

on graphical representations. It presupposes high-precision construction of the curves $\gamma(\lambda)$ and $\lambda(\gamma)$ by an accelerated convergence method using the residual of the solutions of the boundary-value problem corresponding to the abscissa [14]. At the initial step, one in some way determines whether boundary-value problem (1.2) or (2.2) has a solution and estimates the domains (e.g. intervals) of parameter values $\gamma \in \Gamma$, $\lambda \in \Lambda$, for which this solution exists. One then constructs the functions $\gamma = \gamma(\lambda)$ and $\lambda = \lambda(\gamma)$ for the corresponding domains of variation of the arguments (see Fig. 3).

We will first present the scheme for constructing the function $\gamma(\lambda)$ to as high on accuracy as desired for all $\lambda \in \Lambda$, in particular, $\lambda_0 = \lambda^*$, where λ^* is the estimate for λ described in Section 2. The solution of boundary-value problem (2.2) for y for a fixed value of $\lambda = \lambda_0$ will be constructed by the rapidly converging accelerated convergence method described in [14]

$$\gamma^{(n+1)} = \gamma^{(n)} + \varepsilon^{(n)} l K_0^{(n)}, \quad \varepsilon^{(n)} = 1 - \frac{\xi^{(n)}}{l}$$

$$\xi^{(n)} = \xi^{(n)}(\lambda_0) = \arg_x Y(x, \gamma^{(n)}, \lambda_0)$$

$$K_0^{(n)} = -\frac{Y'(\xi^{(n)}, \gamma^{(n)}, \lambda_0)}{V(\xi^{(n)}, \gamma^{(n)}, \lambda_0)} = -\frac{Y'(l, \gamma^{(n)}, \lambda_0)}{V(l, \gamma^{(n)}, \lambda_0)} + O(\varepsilon^{(n)}), \quad V(l, \gamma^*, \lambda_0) \neq 0 \tag{3.1}$$

$$\gamma^{(0)} = \gamma^*, \quad \xi^{(0)}(\lambda_0) = \arg_x Y(x, \gamma^*, \lambda_0)$$

$$\varepsilon^{(0)}(\lambda_0) = \varepsilon, \quad |\varepsilon| \ll 1; \quad n = 0, 1, 2, \ldots$$

$$\varepsilon^{(n)} \leqslant d(\varepsilon^{(n-1)})^2 \leqslant \ldots \leqslant d^{-1}(d\varepsilon)^{\theta(n)}, \quad \theta(n) = 2^n, \quad d(\lambda_0) = \text{const}$$

The desired solution y, y' is determined with an error $O(\varepsilon^{(n+1)})$ on the basis of the functions $Y(x, \gamma^{(n)}, \lambda_0)$ and $Y = dY/dx$. As is shown by computational experience and a theoretical estimation of the accuracy, the solution procedure (3.1) for one-dimensional boundary-value problem (2.2) for y, for a fixed value of $\lambda = \lambda_0$ ($\delta\lambda = 0$) and subject to successful choice of the estimate γ^* ($\varepsilon d \sim 0.1$), yields the desired quantities after two or three iterations, with a relative error of the order of 10^{-4}–10^{-8}, which is sufficient for practical applications. It is usually pointless to increase the number of iterations, because

of the accumulation of rounding errors. The expression for the increment to $\gamma^{(n+1)}$ (3.1) may be expressed in terms of the residual corresponding to y, as in the standard Newton-method approach (see Section 2). When that is done there is no need to carry out the operation of determining the abscissa $\xi^{(n)}(\lambda_0)$ with a relative error $O(\varepsilon^{(n)2}) = O(\varepsilon^{(n+1)})$. However, this operation seems useful and important when solving complex non-linear problems, in order to obtain a confident idea of the existence of a solution and to analyse the behaviour of the residual corresponding to the abscissa [14].

Next, using the procedure of continuation with respect to the parameter λ, one chooses a sufficiently close value of $\lambda = \lambda_1 = \lambda_0 + \Delta\lambda_1$, taking $\gamma^{(0)}(\lambda_1) = \gamma^{(n)}(\lambda_0) \approx \gamma(\lambda_0)$ as the initial approximation. The desired solution with the required accuracy is constructed by the recurrence algorithm (3.1): $\gamma^{(n)}(\lambda_1) \approx \gamma(\lambda_1)$, and so on. In the following steps pertaining to the parameter λ, one can apply extrapolation (linear, quadratic or cubic) of the initial approximation $\gamma^{(0)}(\lambda_i)$, $\lambda_i \in \Lambda (i = 0, 1, \ldots)$ to accelerate the convergence. Thus, the curve $\gamma_y(\lambda)$, $\lambda \in \Lambda$ will be constructed with the required accuracy by solving boundary-value problem (2.2) for y.

Note that, according to relations (3.1), the sensitivity coefficient $V(x, \gamma, \lambda)$ of the solution $Y(x, \gamma, \lambda)$ must be non-zero in a small ε-neighbourhood of the point $x = l$ for the values of $\gamma \in \Gamma$ and $\lambda \in \Lambda$ under consideration. This ensures that the functions $\gamma_y(\lambda)$ will be single-valued and smooth. Otherwise, i.e. in a domain where $V \approx 0$, the inverse function $\lambda_y(\gamma)$ is constructed (see the discussion below and Fig. 3).

By means of an analogous recurrence algorithm, the curve $\lambda = \lambda_z(\gamma)$, $\gamma \in \Gamma$, is determined by solving boundary-value problem (2.2) for Z at fixed values of the parameter γ. As before, a certain parameter value γ_0 is specified, in particular, $\gamma_0 = \gamma^* \in \Gamma$, where γ^* is the estimate $\Delta\gamma = 0$ (see Section 2). An analogous construction, using the rapidly converging accelerated convergence method, yields a solution with the desired accuracy

$$\lambda^{(m+1)} = \lambda^{(m)} + \mu^{(m)} l \Gamma_0^{(m)}, \quad \mu^{(m)} = 1 - \frac{\zeta^{(m)}}{l}$$

$$\zeta^{(m)} = \zeta^{(m)}(\gamma_0) = \arg_x Z(x, \gamma_0, \lambda^{(m)})$$

$$\Gamma_0^{(m)}(\gamma_0) = -\frac{N(\zeta^{(m)}, Y(\zeta^{(m)}, \gamma_0, \lambda^{(m)}), Y'(\zeta^{(m)}, \gamma_0, \lambda^{(m)}))}{S(\zeta^{(m)}, \gamma_0, \lambda^{(m)})} =$$

$$= -\frac{N(l, Y(l, \gamma_0, \lambda^{(m)}), Y'(l, Y(l, \gamma_0, \lambda^{(m)}))}{S(l, \gamma_0, \lambda^{(m)})} + O(\mu^{(m)2}) \tag{3.2}$$

$$S(l, \gamma_0, \lambda^*) \neq 0, \quad Z' = N$$

$$\lambda^{(0)} = \lambda^*, \quad \zeta^{(0)}(\gamma_0) = \arg_x Z(x, \gamma_0, \lambda^*)$$

$$\mu^{(0)}(\lambda_0) = \mu, \quad |\mu| \ll 1, \quad m = 0, 1, 2, \ldots$$

$$\mu^{(m)} \leqslant h(\mu^{(m-1)})^2 \leqslant \ldots \leqslant h^{-1}(h\mu)^{\theta(m)}, \quad \theta(m) = 2^m, \quad h(\gamma_0) = \text{const}$$

The solution of boundary-value problem (2.2) for z is determined with an error $O(\mu^{(m+1)})$ in the ordinate and the abscissa. The closeness of $Y(x, \gamma_0, \lambda^{(m)})$ to the solution of the boundary-value problem is determined by the quantity ε, that is, by the closeness of γ_0 to the exact value. As in the case of (3.1), the convergence of procedure (3.2) is quadratic with respect to the parameter μ (or, more precisely, μh). We also note that he correction to $\lambda^{(m+1)}$ in (3.2) may be expressed in terms of the residual corresponding to Z (see above). When that is done, however, one has no control over the residual corresponding to the abscissa or over the satisfaction of the boundary condition.

Continuing along the same lines as before, one now uses the procedure of continuation with respect to the parameter $\gamma = \gamma_1 = \gamma_0 + \Delta\gamma_1$, where $\Delta\gamma_1$ is sufficiently small, taking $\lambda^{(0)}(\gamma_1) = \lambda^{(m)}(\gamma_0)$ as the initial approximation, and so on. The argument $\gamma \in \Gamma$ takes a sequence of consecutive values $\gamma = \gamma_j$ ($j = 0$, 1, ...), which are enough to construct the curve $\lambda_z(\gamma)$ with the desired accuracy. Acceleration of the convergence and an increase on the stepsize $\delta\gamma$ are achieved by extrapolation of a suitable order. As a result one constructs the curve $\lambda_z(\gamma)$, $\gamma \in \Gamma$, with accuracy sufficient for subsequent computations. Here it is essential to assume that the sensitivity coefficient $S(l, \gamma, \lambda)$ of the solution $Z(x, \gamma, \lambda)$ does not vanish in a small neighbourhood of the point $x = l$ for the values of $\gamma \in \Gamma$ and $\lambda \in \Lambda$ under consideration. This ensures that the function $\lambda_z(\gamma)$ will be single-valued and smooth; otherwise, one constructs the inverse function $\gamma_z(\lambda)$ (see below and Fig. 3).

After numerical or graphical construction of the functions $\gamma_z(\lambda)$ and $\lambda_z(\gamma)$, one finds the common point (γ^0, λ^0) as the point at which these curves intersect. It determines the required solution of boundary-value problem (2.2) and is a root of the system

$$\gamma = \gamma_y(\lambda), \quad \lambda = \lambda_z(\gamma), \quad \gamma^0 = \gamma_y(\lambda_z(\gamma^0)), \quad \lambda^0 = \lambda_z(\gamma^0). \tag{3.3}$$

This point is found numerically by polynomial interpolation of the functions $\gamma_y(\lambda_i)$ and $\lambda_z(\gamma_i)$ in a neighbourhood of the point of intersection (e.g., linear interpolation).

Note that in a sufficiently small neighbourhood of the required root (γ^0, λ^0) one can use a recurrence procedure to pass from the curve $\gamma = \gamma_y(\lambda)$, from a point $\gamma_i = \gamma_y(\lambda_i)$ on the curve $\lambda = \lambda_z(\gamma)$, to the point $\lambda_{i,1} = \lambda_z(\gamma_y, (\lambda_i)) = \lambda_z(\gamma_i)$, and then again on the curve $\gamma = \gamma_y(\lambda)$ to the point $\gamma_{i,1}$; and so on (see the neighbourhood of the point A in Fig. 3). The process will converge like a geometric progression to the required point with exponent χ, $0 < \chi < 1$, if the bisector of the angle at which the curves $\lambda_z(\gamma)$ and $\gamma_y(\lambda)$ intersect (shown in Fig. 3 by the dot dash line) is inclined to the left of the vertical (a positive angle); the convergence will be of the type of a stable focus or node. Otherwise the process will not converge. If the angle is negative (an unstable focus or node, point B in Fig. 3), the inverse procedure will lead to convergence: passage from the curve $\lambda = \lambda_z(\gamma)$ from a point $\lambda_j = \lambda_z(\gamma_j)$ on the curve $\gamma = \gamma_y(\lambda)$ to the point $\gamma_{j,1} = \gamma_y(\lambda_z(\gamma_j)) = \gamma_y(\lambda_j)$, and then again on the curve $\lambda = \lambda_z(\gamma)$ to the point $\lambda_{j,1}$; and so on. In the critical case of zero angle (point or centre), use of the secant method will yield quadratic convergence with respect to a small parameter, representing the closeness of the points $(\gamma_i = \lambda_j)$ and $(\gamma_{i+1} = \lambda_{j+1})$ to the point (γ^0, λ^0) (point C in Fig. 3). Knowing the three points $(\gamma_{i-1}, \lambda_{i-1})$, (γ_i, λ_i) and $(\gamma_{i+1}, \lambda_{i+1})$, one can apply the method of parabolas. This approach may also be used in the previous cases of convergent (point A in Fig. 3) or divergent (point B in Fig. 3) iterative processes. The critical case, when the curves (3.3) coincide (point D in Fig. 3) requires a separate consideration.

As noted previously, when constructing the curves $\gamma = \gamma_y(\lambda)$ and $\lambda = \lambda_z(\gamma)$ it is essential to assume that $V(l, \gamma, \lambda)$ and $S(l, \gamma, \lambda)$ do not vanish in some neighbourhood of $x = l$ and $\gamma \in \Gamma$, $\lambda \in \Lambda$. Otherwise, the corresponding corrections to $\gamma^{(n)}$ and $\lambda^{(n)}$ will not necessarily be small, nor will the curves be single-valued and have bounded derivatives. In the regions $\gamma \in \Gamma_0 \subseteq \Gamma$ and $\lambda \in \Lambda_0 \subseteq \Lambda$ where this happens, it makes sense to construct the inverse functions $\lambda = \lambda_y(\gamma)$ and $\gamma = \gamma_z(\lambda)$ (see above).

To fix our ideas, let us consider the construction of the curve $\lambda = \lambda_y(\gamma)$, $\gamma \in \Gamma_0$. According to relations (2.1)–(2.3), the boundary-value problem of y, considered at a fixed value of $\gamma_i \in \Gamma_0$, $\delta\gamma = 0$, yields recurrence relations for λ analogous to (3.1)

$$\lambda^{(n+1)} = \lambda^{(n)} + \varepsilon^{(n)} l \Pi_i^{(n)}, \quad \varepsilon^{(n)} = 1 - \frac{\xi^{(n)}}{l}$$

$$\xi^{(n)} = \xi^{(n)}(\gamma_i) = \arg_x Y(x, \gamma_i, \lambda^{(n)})$$

$$\Pi_i^{(n)} = -\frac{Y'(\xi^{(n)}, \gamma_i, \lambda^{(n)})}{U(\xi^{(n)}, \gamma_i, \lambda^{(n)})} = -\frac{Y'(l, \gamma_i, \lambda^{(n)})}{U(l, \gamma_i, \lambda^{(n)})} + O(\varepsilon^{(n)}), \quad U(l, \gamma, \lambda) \neq 0 \tag{3.4}$$

$$\lambda^{(0)} = \lambda^*, \quad \xi^{(0)} = \arg_x Y(x, \gamma, \lambda^{(0)}), \quad \varepsilon^{(0)} = \varepsilon, \quad |\varepsilon| \ll 1$$

The assumption that the sensitivity coefficient $U = \partial Y/\partial\lambda \neq 0$ when $x = l$, is quite natural. Otherwise, the solution of boundary-value problem (2.2) for y will be locally independent of the parameters γ and λ, which is an extremely degenerate situation. Thus, relations (3.4) yield the construction of a curve $\lambda = \lambda_y(\gamma)$ (or part of the curve contiguous with part of the curve $\gamma = \gamma_y(\lambda)$) which, together with the curve $\lambda = \lambda_z(\gamma)$, determines the required point (γ^0, λ^0) (see above).

We will also briefly describe an algorithm for constructing the curve $\gamma = \gamma_z(\lambda)$ (or part of it) for the case when $S(l, \gamma, \lambda)$ is small in the neighbourhood of $x = l$, $\lambda_j \in \Lambda_0$, under consideration. According to (2.1)–(2.3), the boundary-value problem for z yields, for fixed λ_j, $\delta\lambda = 0$, recurrence relations analogous to (3.2) for γ

$$\gamma^{(m+1)} = \gamma^{(m)} + \mu^{(m)} l D_0^{(m)}, \quad \mu^{(m)} = 1 - \frac{\zeta^{(m)}}{l}$$

$$\zeta^{(m)} = \zeta^{(m)}(\lambda_j) = \arg_x Z(x, \gamma^{(m)}, \lambda_j) \tag{3.5}$$

$$D_j^{(m)} = -\frac{Z(\zeta^{(m)}, \gamma^{(m)}, \lambda_j)}{W(\zeta^{(m)}, \gamma^{(m)}, \lambda_j)} = -\frac{Z(l, \gamma^{(m)}, \lambda_j)}{W(l, \gamma^{(m)}, \lambda_j)} + O(\mu^{(m)})$$

$$W(l, \gamma, \lambda_j) \neq 0, \quad \gamma^{(0)} = \gamma^*, \quad \mu^{(0)} = \mu, \quad |\mu| \ll 1, \quad \zeta^{(0)} = \arg_x Z(x, \gamma^*, \lambda_j)$$

We recall that, according to relations (2.2), we have $Z' = N$. Consequently, algorithm (3.5) and the procedure of continuation with respect to the parameter $\lambda = \lambda_j \in \Lambda_0$ yield a construction of the required curve $\lambda = \lambda_z(\gamma)$ which, together with the curve $\gamma = \gamma_y(\lambda)$, determines the required point (γ^0, λ^0). The assumption that $W \neq 0$ in the neighbourhood of the x, γ and λ values under consideration is natural, since otherwise one again obtains a very degenerate situation: the solution of the boundary-value problem for Z will be locally independent of γ and λ.

As remarked previously, these algorithms may be expressed in a standard form (see Section 2) which does not require the determination of the residuals of the solutions to the Cauchy problems for Y and Z corresponding to x. This approach, however, does not ensure convergence of the computation process when one is solving essentially non-linear boundary-value problems, because of the different behaviour of each of the variables y and z and the different accuracies with which they are computed. Consideration of the residual corresponding to the abscissa enables one to stabilize the computation process and to monitor that the boundary conditions are satisfied; it does not slow down the process in practice.

4. MODIFICATION OF THE ACCELERATED CONVERGENCE METHOD

Other boundary conditions. For solving applied problems, other types of boundary conditions are of interest. For example, at one or both endpoints the variable y must satisfy conditions "of the second kind": $y'(0) = 0$ and/or $y'(l) = 0$ (see (1.1)). We will first present an accelerated convergence algorithm for the case $y'(0) = 0, y'(l) = 0$. Then the conditions for the new sensitivity function $v = \partial y/\partial\theta$ in relations (2.1) have the form $v(0) = 1$, $v'(0) = 0$, where we have introduced a parameter $\theta \in \Theta$, to be determined from boundary-value problem (2.2) for y with conditions $y(0) = \theta, y'(0) = 0$. The sensitivity function $w = \partial Z/\partial\theta$ and the other functions are defined as before. Equations (2.3)–(2.8) are considered after changing the notation $\gamma \to \theta$, $\Delta\gamma \to \Delta\theta$.

The algorithms for constructing the curves $\theta = \theta_y(\lambda)$, $\lambda = \lambda_z(\theta)$ and/or $\lambda = \lambda_y(\theta)$, $\theta = \theta_z(\lambda)$ have the form of (3.1), (3.2) and/or (3.4), (3.5), with the above-mentioned changes of notation: $\gamma \to \theta \in \Theta$ (and/or $\theta \in \Theta_0 \subseteq \Theta$), $\Delta\gamma_j \to \Delta\theta_j$ (and/or $\Delta\gamma_i \to \Delta\theta_i$). Thus, the essential difference in the solution of the boundary-value problem is the change in the initial data when integrating the Cauchy problem for Y and V^*.

The case of boundary conditions of the form $y(0) = y'(l) = 0$ can be reduced to that considered above by introducing a new argument $\chi = l - x$ and changing the notation of the functions F and N in relations (1.1) and (1.2), and so on.

But if the conditions are imposed at both endpoints, $y'(0) = y'(l) = 0$, the algorithms presented in Sections 2 and 3 must be modified. As before, γ is replaced by an unknown parameter θ and the boundary condition for determining it takes the form $y'(l, \theta, \lambda) = 0$. In the standard algorithm (2.3)–(2.9) one applies the transformations $V \to V'$, $U \to U'$. Algorithms (3.1), (3.2), (3.4), (3.5) for constructing curves $\theta_{y,z}(\lambda)$ and $\lambda_{z,y}(\theta)$ are modified as follows:

$$K_i^{(n)} = -\frac{Y''(\xi^{(n)}, \theta^{(n)}, \lambda_i)}{V'(\xi^{(n)}, \theta^{(n)}, \lambda_i)}, \quad \xi^{(n)}(\lambda_i) = \arg_x Y'(x, \theta^{(n)}, \lambda_i)$$

$$\Gamma_j^{(m)} = -\frac{Z'(\zeta^{(m)}, \theta_j, \lambda^{(m)})}{S(\zeta^{(m)}, \theta_j, \lambda^{(m)})}, \quad \zeta^{(m)}(\theta_j) = \arg_x Z(x, \theta_j, \lambda^{(m)}) \tag{4.1}$$

$$\Pi_i^{(n)} = -\frac{Y''(\xi^{(n)}, \theta_i \lambda^{(n)})}{U'(\xi^{(n)}, \theta_i, \lambda^{(n)})}, \quad \xi^{(n)}(\theta_i) = \arg_x Y'(x, \theta_i, \lambda^{(n)})$$

$$D_j^{(m)} = -\frac{Z'(\zeta^{(m)}, \theta^{(m)}, \lambda_j)}{W(\zeta^{(m)}, \theta^{(m)}, \lambda_j)}, \quad \zeta^{(m)}(\lambda_j) = \arg_x Z(x, \theta^{(m)}, \lambda_j)$$

Thus, according to algorithms (4.1), the main difference lies in the definition of the residuals corresponding to x and the definition of the coefficients of the corrections. In addition, the Cauchy problem for Y and V is formulated differently.

Note that the algorithms may be modified in the case of more-general boundary conditions, "of the third kind"

$$[\alpha(x)y+\beta(x)y']_{x=0,l}=0, \quad [\alpha^2+\beta^2]_{x=0,l}\neq 0 \tag{4.2}$$

The expressions obtained are extremely lengthy and will therefore not be given here.

For applications, it may be of interest to consider the case of non-linear boundary conditions generalizing (4.2)

$$\varphi(x,y,y')|_{x=0,l}=0, \quad [\varphi_y'^2+\varphi_{y'}'^2]_{x=0,l}\neq 0 \tag{4.3}$$

The algorithms for solving boundary-value problem (1.2) with conditions (4.3) may also be modified in a suitable way and expressed as recurrence computation procedures. In their general form, however, they are quite lengthy expressions, which are only worth presenting for specific functions F, N and φ.

A combination of the accelerated convergence and penalty methods. For large-scale applied calculations, it may be desirable, before implementing the investigations described in Section 3, to use a penalty method combined with accelerated convergence based on a computation and analysis of residuals corresponding to the abscissa. Instead of demanding strict satisfaction of the boundary conditions for the variables Y and Z, the magnitude of the residual is minimized at each step of the iteration using a penalty method [3–5], e.g. in the form

$$\Xi^2(x)=c_Y^2Y^2(x,\gamma,\lambda)+c_Z^2Z^2(x,\gamma,\lambda)\to \min_x \tag{4.4}$$

where c_Y^2 and c_Z^2 are weighting factors, yet to be chosen.

Let us analyse the behaviour of the quantity Ξ^2 near $x=l$ and values of γ and λ close to the required γ^0 and λ^0. We note that when $\gamma=\gamma^0$, $\lambda=\lambda^0$ there is an absolute minimum at $x=l$. The variables Y and Z then admit of the following approximate representations

$$Y(x)\approx Y(l)+Y'(l)(x-l), \quad Z(x)\approx Z(l)+Z'(l)(x-l) \tag{4.5}$$

Here and below, the dependence on the parameters γ and λ is not indicated, for brevity.

Substituting expressions (4.5) into (4.4), we carry out the elementary operation of minimizing the function $\Xi^2(x)$ with respect to x. The following relations are obtained for the optimal value $x=\eta$

$$\begin{aligned}
&\eta-l=-d^{-1}(c_Y^2Y(l)Y'(l)+c_Z^2Z(l)Z'(l))\\
&\Xi^2(\eta)=E_Y^2(\eta)+E_Z^2(\eta)=c_Y^2c_Z^2d^{-1}G^2\\
&d=c_Y^2Y'^2(l)+c_Z^2Z'^2(l), \quad G=Y(l)Z'(l)-Y'(l)Z(l)\\
&E_Y^2=c_Y^2c_Z^4Z'^2(l)d^{-2}G^2, \quad E_Z^2=c_Z^2c_Y^4Y'^2(l)d^{-2}G^2
\end{aligned} \tag{4.6}$$

The weighting factors $c_{Y,Z}^2$ may be chosen taking into account the relative importance of the boundary conditions for y (the solution of the initial boundary-value problem) and for Z (the satisfaction of the isoperimetric conditions) for solving the boundary-value problem. Suppose the ratio of these residuals is taken to be $v^2=E_Y^2/E_Z^2$; it then follows from (4.6) that

$$\begin{aligned}
&c_Y^2=\frac{\nu^2}{Y'^2(l)}\approx\frac{\varepsilon^2l^2\nu^2}{Y^2(l)}, \quad Y(l)\neq 0\\
&c_Z^2=\frac{1}{Z'^2(l)}\approx\frac{\mu^2l^2}{Z^2(l)}, \quad Z(l)\neq 0
\end{aligned} \tag{4.7}$$

where the parameters ε and μ are defined as in (3.1) and (3.2). The coefficients c_Y^2 and c_Z^2 may be made more accurate at each step of the iteration of an appropriate recurrence accelerated convergence algorithm.

This algorithm will now be described. It is constructed using the sensitivity functions $q = \partial y/\partial l$ and $r = \partial z/\partial l$, where

$$q = \gamma' v + \lambda' u, \quad r = \gamma' w + \lambda' s, \quad \gamma = \gamma(l), \quad \lambda = \lambda(l)$$
$$q(l) = -y'(l), \quad r(l) = -z'(l) = -N(l, 0, y'(l)) \tag{4.8}$$

The primes denote differentiation with respect to l. The unknown parameters $\gamma(l)$ and $\lambda(l)$ are made more accurate by successive approximations following the scheme

$$\gamma^{(n+1)} = \gamma^{(n)} + \delta^{(n)} l K^{(n)}, \quad K^{(n)} = -\frac{Y'_{(n)}(l) S_{(n)}(l) - Z'_{(n)}(l) U_{(n)}(l)}{\Delta_{(n)}(l)}$$
$$\lambda^{(n+1)} = \lambda^{(n)} + \delta^{(n)} l \Gamma^{(n)}, \quad \Gamma^{(n)} = \frac{Y'_{(n)}(l) W_{(n)}(l) - Z'_{(n)}(l) V_{(n)}(l)}{\Delta_{(n)}(l)} \tag{4.9}$$
$$\delta^{(n)} = 1 - \frac{\eta^{(n)}}{l}, \quad \eta^{(n)} = \arg_x [c_Y^2 Y_{(n)}(x) Y'_{(n)}(x) + c_Z^2 Z_{(n)}(x) Z'_{(n)}(x)]$$
$$\frac{d^2 E^2}{d\eta^{(n)2}} \neq 0$$

The determinant $\Delta_{(n)}$ has the form of (2.8); it is evaluated on the basis of the values of the sensitivity coefficients in (4.8). The functions $Y_{(n)}(x)$, $Z_{(n)}(x)$, $V_{(n)}(x)$, $U_{(n)}(x)$, $W_{(n)}(x)$, $S_{(n)}(x)$ also depend on the parameters $\gamma^{(n)}$, $\lambda^{(n)}$ computed at the previous step of the iterations. For brevity, this dependence is not indicated explicitly.

It should be noted that algorithm (4.9) possesses accelerated convergence for the criterion E^2 relative to the initial small parameter $\delta = \delta(\gamma^*, \lambda^*)$

$$\delta^{(n)} \leqslant \alpha(\delta^{(n-1)})^2 \leqslant \ldots \leqslant \alpha^{-1}(\alpha\delta)^{\theta(n)}, \quad \theta(n) = 2^n$$
$$\eta^*(\gamma^*, \lambda^*) = \arg_x [c_Y^2 Y(x, \gamma^*, \lambda^*) Y(x, \gamma^*, \lambda^*) + c_Z^2 Z(x, \gamma^*, \lambda^*) Z'(x, \gamma^*, \lambda^*)]$$
$$\delta = 1 - \eta/l, \quad |\delta| \leqslant 1, \quad \alpha = \text{const} \tag{4.10}$$

It is assumed that η is a simple root of the equation $E^{2'} = 0$. An algorithm analogous to (4.9), (4.10) may also be proposed for the more general boundary conditions (4.2) and (4.3). In that case appropriate changes must be made in the criterion E^2, characterizing the residual in the boundary conditions.

The essential distinction of algorithm (4.4)–(4.10) consists in the minimization of the residual $E^2(x)$ as a function of x for known approximate values of the parameters γ and λ, with subsequent improvement of their accuracy. In the standard algorithm (see Section 2 and (2.9)), $E^2(\gamma, \lambda)$ is minimized with respect to γ and λ at a fixed value of $x = l$; on the basis of this minimization one then improves the accuracy of γ and λ, and this procedure, as we known, may lead to divergence of the recurrence procedure owing to "ravine effects".

5. COMPUTATION OF THE STEADY INCOMPRESSIBLE VISCOUS FLOW IN A PLANE CONVERGENT CHANNEL (JEFFREY-HAMEL PROBLEM)

After changing to the normalized argument x and function $y(x)$, the equations (see [15]) of the nonlinear boundary-value problem for the velocity profile, and the conditions for the flow rate to be constant can be written in the following form [16]

$$y'' + a^2 y - b y^2 = \lambda, \quad y(0) = y(1) = 0, \quad a = 4\beta, \quad b = 2\beta \,\mathrm{Re}$$
$$z' = y - 1, \quad z(0) = z(1) = 0, \quad 0 \leqslant \beta \leqslant \pi, \quad \mathrm{Re} > 0, \quad p = (\lambda - a^2 y)/(a^2 b) \tag{5.1}$$

where 2β is the aperture angle, Re is the Reynolds number and p is the pressure profile. The arbitrary

constant λ has the meaning of a Lagrange multiplier in the isoperimetric variational problem (1.1) corresponding to (5.1), with functions

$$F = \frac{1}{2}\left(y'^2 - a^2y^2 + \frac{2}{3}by^3\right), \quad N = y - 1 \tag{5.2}$$

When Re = 0, problem (5.1) has no mechanical content; nevertheless, its solution is useful as a first approximation in the procedure of the perturbation method for sufficiently small values of Re > 0, subsequently carrying out continuation with respect to the parameter Re at a fixed value of a [16]. The case $a = 0$ corresponds to Poiseuille flow [15]; if $a \ll 1$ and Re ~ 1, an approximate solution is constructed by expansions in powers of the parameter a. In the case $b \gg 1$ one can construct [15] an asymptotic first approximation $y^{(1)}(x)$, from which it follows that $\gamma^{(1)} = y^{(1)\prime}(0) = \sqrt{4b/3}$, $\lambda^{(1)}(0) = y^{(1)\prime\prime}(0) = -b$ [16]. The solution of problem (5.1) is used as a generating solution to investigate more complex processes: heat-mass transfer, viscoelastic flow, etc.

In applications one is particularly interested in flows with moderately large Reynolds numbers Re at comparatively small aperture angles 2β, so that the need arises for computational algorithms. For values of practical importance Re ~ 0.1–10^3, β ~ 0.1, in particular, $\beta^\circ = 5^\circ$–10°, effective numerical–analytical methods need to be developed for solving problem (5.1). The modified Newton-type accelerated convergence method developed in Sections 2–4, combined with the procedure of continuation with respect to the parameters, enables one, after one or two iterations, using relatively low computational resources, to obtain extremely accurate values of the unknown quantities. As a result one obtains a virtually exact solution with a relative error of 10^{-5}–10^{-7}, which can even be reduced if necessary. The algorithm of the method involves computing the missing quantities $\gamma = y'(0)$ and $\lambda = y''(0)$ which determine the functions $y(x)$ and $z(x)$ as a solution of Cauchy problem (5.1) at fixed a and b, i.e. β and Re.

1. Let us first consider the simplest (classical) case of single-mode flow, symmetrical about $\theta = 0$ ($x = 1/2$) [15]. It is assumed at the initial step of the algorithm that sufficiently accurate estimates γ_0 and λ_0 are available for γ and λ. They may be found by variational techniques on the basis of functionals (1.1), (5.2) and a suitably selected test function $y^*(x)$ [14, 16]. As computations have shown, the function $y(x, a, b)$ is fairly simple in form. The procedure of continuation with respect to the parameters β and Re (or a and b) is extremely convenient. In particular, for fixed $a = 4\beta > 0$ it is proposed to take

$$\gamma_0 = 8\beta^2 l \sin 2\beta, \quad \lambda_0 = -32\beta^3 l \cos 2\beta$$

as the initial approximations of $\gamma_0(b_1)$ and $\lambda_0(b_1)$ for sufficiently small $b_1 > 0$. Using the recurrence accelerated convergence algorithm (see Sections 2 and 3) with the required accuracy one determines $\gamma_1 = \gamma(b_1)$ and $\lambda_1 = \lambda(b_1)$, which are taken as initial approximations for $\gamma_0(b_2)$ and $\lambda_0(b_2)$; and so on. The existence of sets $\{\gamma_k\}$ and $\{\lambda_k\}$ of sufficiently accurate values of γ_k and λ_k ($k = 1, 2, \ldots$) enables one to extend the interval $\delta b_{k+1} = b_{k+1} - b_k$ by polynomial (usually linear or quadratic) extrapolation of the initial approximations $\gamma_0(b_{k+1})$ and $\lambda_0(b_{k+1})$.

The iterative algorithm for improving the approximations to the required quantities $\gamma(b)$ and $\lambda(b)$ at the first and following steps involves integrating two Cauchy problems for sixth-order systems or one Cauchy problem for a ninth-order system. These problems are described by the simultaneous equations (5.1) for y and z, with the conditions

$$y(0) = z(0) = 0, \quad y'(0) = \gamma_n(b), \quad \lambda = \lambda_n(b), \quad n = 0, 1, \ldots, \quad b \in \{b_k\} \tag{5.3}$$

and equations for the sensitivity functions v, w and u, s, that is, the derivatives of the solution $y(x)$, $z(x)$ with respect to γ and λ, respectively

$$\begin{gathered} v'' + a^2 v - 2byv = 0, \quad w' = v, \quad v(0) = w(0) = 0, \quad v(0) = 1 \\ u'' + a^2 u - 2byu = 1, \quad s' = u, \quad u(0) = u'(0) = s(0) = 0 \end{gathered} \tag{5.4}$$

The Cauchy problems (5.1), (5.3) and (5.4) are integrated simultaneously or separately and the functions $y_n(x)$ and $z_n(x)$ are determined. To refine the quantities $\gamma_n(b)$ and $\lambda_n(b)$ one needs to know the functions y_n, z_n, v_n, w_n, u_n and s_n at the final point $x = 1$ for $b \in \{b_k\}$. The standard scheme of Newton's method has the form (see Section 2)

$$\gamma_{n+1}(b)=\gamma_n(b)+\delta\gamma_n(b),\quad \lambda_{n+1}(b)=\lambda_n(b)+\delta\lambda_n(b)$$
$$\delta\gamma_n=-[y_n(1)s_n(1)-z_n(1)u_n(1)]\Delta_n^{-1}(1)$$
$$\delta\lambda_n=[y_n(1)w_n(1)-z_n(1)v_n(1)]\Delta_n^{-1}(1);\quad n=0,1,\ldots \tag{5.5}$$
$$\Delta_n(x)=v_n(x)s_n(x)-u_n(x)w_n(x),\quad \Delta_n(1)\neq 0$$

The recurrence n-process (5.5) is continued until the required accuracy is achieved, that is, until the residuals $y_n(1)$ and $z_n(1)$ are sufficiently small. Convergence of the algorithm requires thorough verification at each step and depends on te values of the residuals and the determinant $\Delta_n(1)$ of the matrix of the sensitivity coefficients. A specific feature of the proposed modification of the accelerated convergence method is the additional verification that the process is convergent according to the abscissae ξ_n and η_n, that is, according to the quantities $|\varepsilon_n|$ and $|\mu_n|$

$$\varepsilon_n=1-\xi_n,\quad \mu_n=1-\eta_n;\quad \xi_n=\arg y_n(x),\quad \eta_n=\arg z_n(x) \tag{5.6}$$

The values of ξ_n and η_n in (5.6) are the roots (zeros) closest to $x = 1$ of the functions $y_n(x)$ and $z_n(x)$. The effectively verifiable conditions $|\varepsilon_0|, |\mu_0| \ll 1, |\varepsilon_n|, |\mu_n| \to 0$ indicate the existence of the required solution of boundary-value problem (5.1) and the convergence of algorithm (5.1), (5.3)–(5.6), which is accelerated (quadratic); the quantities ε_n and μ_n of (5.6) satisfy the estimates

$$|\varepsilon_{n+1}|\sim\varepsilon_n^2,\quad |\mu_{n+1}|\sim\mu_n^2,\quad y_n(1)=-\gamma_n\varepsilon_n,\quad z_n(1)=-\mu_n;\quad n=0,1,\ldots \tag{5.7}$$

Convergence depends on the moduli of $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, being small, that is, on $x = 1$ being close to the abscissae ξ_0 and η_0, which is ensured by proper choice of the initial approximation $\gamma_0(b)$, $\lambda_0(b)$, $b \in \{b_k\}$, and by the step-size δb_k or extrapolation. In addition, the condition $|\Delta_n(1)| \geqslant \text{const} > 0$ must be satisfied (see (5.5)). If estimates (5.6) and (5.7) are substantially violated, further analysis is needed, usually associated with degeneracy of the standard algorithm: $\Delta_n(1) \approx 0$ (for example, if $\beta \to \pi/2$ or $a \to 2\pi$). Simultaneous determination of the residuals $y_n(1)$, $z_n(1)$ and ε_n, μ_n enables one to compute highly accurate values of the required parameters $\gamma(b)$ and $\lambda(b)$ at an arbitrary fixed value of a, that is, $0 < \beta < \pi/2$. The procedure of continuation with respect to the parameter a, $a \in \{a_j\}$, is implemented in a similar fashion. If needed, one can construct the functions $\gamma(a, b)$ and $\lambda(a, b)$ by continuation with respect to the parameters a, b ($0 < \beta < \pi$, $0 \leqslant \text{Re} \leqslant \text{Re}^* < \infty$).

Computations have established that the solution of problem (5.1) is obtained in a few iterations (usually just two or three), with an error of 10^{-5}–10^{-7}, without the need to extrapolate with respect to b_k. It should be noted that at the beginning of the calculation (when $b \ll 1$) one has $\Delta_n(1) \approx 0.1$; this requires sufficiently small initial step-sizes $\delta b_k \approx 0.01$ for b. As b increases, the quantity $\Delta_n(1)$ also increases, making it possible almost immediately ($b \sim 0.5$) to enlarge the step-size $\delta b_k \approx 0.1$–1. Computational experience has shown that the smallness of the quantities ε_0 and μ_0 is the decisive factor for rapid convergence of the iterative algorithm; as a rule, it is sufficient that ε_0, $\mu_0 \sim 0.1$.

Using the recurrence algorithm (5.1), (5.3)–(5.5), the required quantities γ and λ and the function $y(x)$ have been computed for fixed values of the angle $\beta^\circ \in [1^\circ$–$89.999^\circ]$; the parameter $b = 2\beta$ Re, $\beta = 2\pi\beta^\circ/360^\circ$ was varied within the limits $b \in [0, 200]$. The computations were carried out with the above-mentioned precision of 10^{-5}–10^{-7} in the residuals $y(1)$ and $z(1)$, and ε and μ. The results for a characteristic value, $\beta^\circ = 10^\circ$, are shown in Figs 4–7, curves 1. The curves $\gamma(b)$ and $\lambda(b)$ are shown in Fig. 4(a) for $1 \leqslant b \leqslant 10$ and in Fig. 4(b) for $10 \leqslant b \leqslant 200$. They defined a solution of Cauchy problem (5.1); the corresponding velocity profiles $y(x)$ for $b = 1$, 10, 200 are shown in Figs 5–7. According to relations (5.1), the pressure $p(x)$ is a linear function of y.

2. Using the algorithm developed above one can effectively investigate multi-mode (asymmetric and symmetric) viscous flows. What makes the construction of multi-mode velocity profiles particularly difficult is the degeneration of the problem as $b \to 0$. This leads to unbounded values of the parameters $\gamma(b)$ and $\lambda(b)$ needed to integrate the appropriate Cauchy problem (5.1). At a certain fixed value of the parameter $b = b_0$ (usually $b_0 \approx 10$) and selected mode level ($n = 2, 3, \ldots$) a search is carried out for the unknown quantities γ and λ and they are computed with high precision, so as to obtain the residual indicated previously corresponding to the boundary conditions. Continuation with respect to the parameter b is then used to construct universal curves $\gamma(b)$ and $\lambda(b)$ for $0 < b < b_0$ and $b_0 < b < \infty$. The computational difficulties are aggravated by the fact that the quantities $\gamma_n \gg \gamma_1$, $\lambda_n \gg \lambda_1$ (by several orders of magnitude) and reach very high values at $b = 1$, e.g., $\gamma_3 \sim 10^4$, $\lambda_3 \sim 10^5$. These circumstances

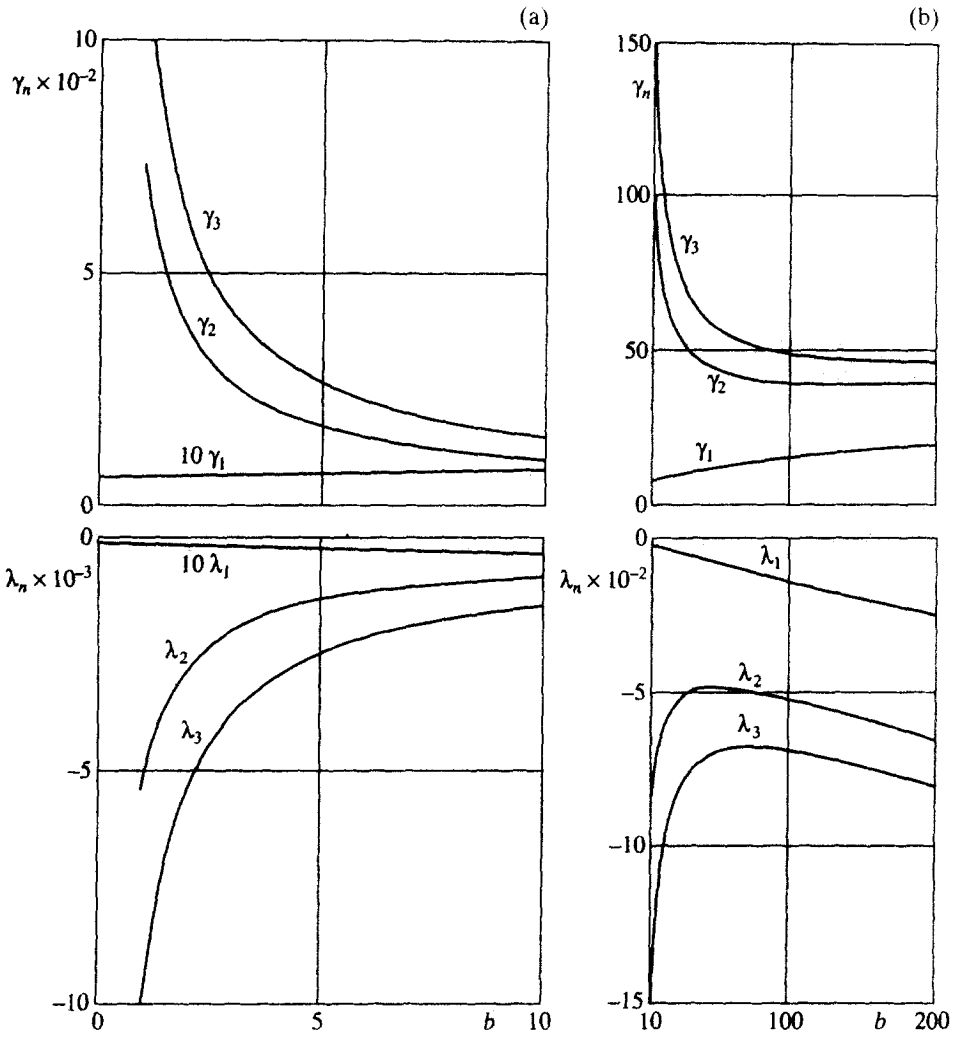
(a)
(b)
$\gamma_n \times 10^{-2}$
γ_n
γ_3
γ_2
$10\,\gamma_1$
γ_1
$\lambda_n \times 10^{-3}$
$\lambda_n \times 10^{-2}$
$10\,\lambda_1$
λ_1
λ_2
λ_3
b

Fig. 4

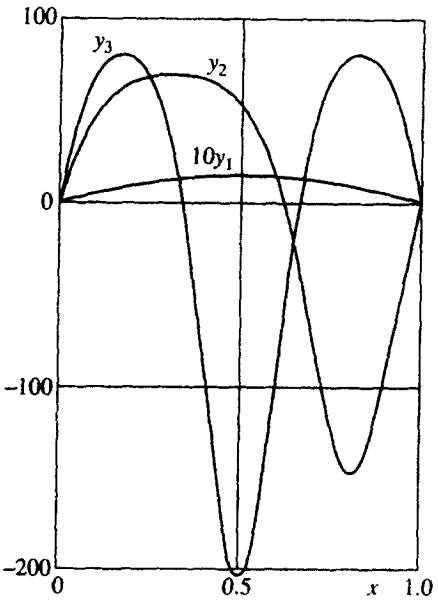
y_3
y_2
$10y_1$
x

Fig. 5

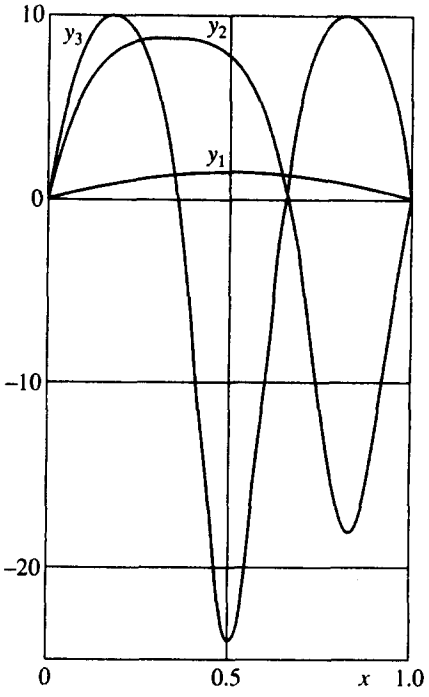

Fig. 6

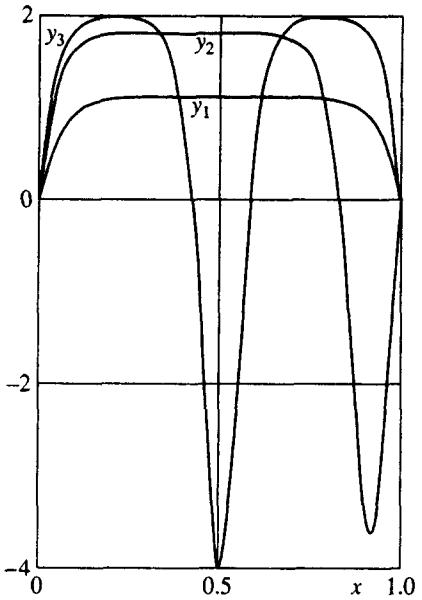

Fig. 7

may explain the fact that, up to the present, no conclusive results have been achieved in determining and analysing multi-mode flows in the Jeffrey-Hamel problem.

A graphical representation of a numerical-analytical investigation of multi-mode flows when $n = 2$, 3, is given in Figs 4–7. They are compared with the curves representing the classical solution ($n = 1$) presented in Section 1, which has been studied previously [16], inter alia for different values of the parameter a. It has been shown that when $n \geqslant 2$ the odd modes n correspond to symmetrical solutions (about $x = {}^1/_2$, i.e. $\theta = 0$) and even modes n correspond to asymmetric solutions.

Analysis indicates that the flows possess well-defined structural properties. Specifically; the positive maxima ($n \geqslant 3$) and negative minima ($n \geqslant 4$) of the functions $y_n(x, b)$ have practically identical values. In addition, at all zeros x_i ($i = 1, \ldots, n+1$) of the function $y_n(x, b)$ (for fixed n and b) the derivatives y_n' are equal in absolute value, that is, $\gamma_n(b) = \pm y_n'(x_i, b)$. Thus, in a certain sense, multi-mode flow is a combination of single-mode and double-mode flows; this obviously follows from the common radiality property of the flows.

Figures 4(a, b) show graphs of the functions $\gamma_n(b)$ and $\lambda_n(b)$ on different scales for $1 \leqslant b \leqslant 10$ and $10 \leqslant b \leqslant 200$, respectively. They define solutions of the boundary-value problem by integrating the Cauchy problems at a fixed value of the parameter a (i.e., the angle β). The curves exhibit interesting properties near $b = 0$, the graphs having vertical asymptotes: $\gamma_n \to +\infty$, $\lambda_n \to -\infty$ as $b \to +0$. We recall that the classical single-mode solution [16] has finite values of γ_1 and λ_1. Also to be noted are the high absolute values of γ_n and λ_n at $b \sim 1$, while the differences $\gamma_n - \gamma_{n-1}$, $\lambda_{n-1} - \lambda_n$ increase without limit as $b \to +0$ and as the number n is increased. Each curve γ_n has a minimum value at a certain fairly high value of $b = b_n^{\gamma} \sim 10^2 - 10^3$ and tends very slowly from above to the asymptote $\gamma^{(1)} = \sqrt{4b/3}$. Similarly, the curves λ_n have maximum values at certain values of $b = b_n^{\lambda} \sim 10^2$ and they too tend very slowly, from below, to an asymptote $\lambda^{(1)} = -b$. This implies a conclusion of importance in the mechanical context: all the steady modes, including the fundamental ($n = 1$, see [16]), tend, in the limit in some metric for $0 < x < 1$, to an ideal flow as $b \to \infty$ (Re $\to \infty$).

In fact, the curves $\gamma_n(b)$ and $\lambda_n(b)$ are the main result of these investigations, and using them, via integration of Cauchy problem (5.1), one obtains the main characteristics of steady flow in a convergent channel: the velocity profile $y(x)$, the pressure $p(x)$, the components of the strain rate tensor and the stress tensor, and so on [16]. The form of the curves γ_n and λ_n is fairly simple, but their construction demands extremely laborious, high-precision, calculations, which become even more involved as $b \to +0$ and $b \to \infty$. The problem exhibits pronounced "boundary-layer" and "ravine" effects. Computational algorithms based on the well-known methods of functional analysis (Bubnov–Galerkin), finite elements, and finite differences do not yield satisfactory results.

Table 1

b	$-p_1^0$	$-p_2^0$	$-p_3^0$	$-\delta_1^* \times 10^5$	$-\delta_2^* \times 10^5$	$-\delta_3^* \times 10^5$
1	26.5219	5.32829	2.55111	5618	2626	2077
10	11105.5	167.974	6.67587	1322	1076	270
200	21817.9	296.093	8.23039	937	807	243

As an illustration of multi-mode flow, Figs 5–7 present velocity profiles $y_n(x)$, $n = 2, 3$, for three values of b: relatively small, $b = 1$; "medium", $b = 10$; and comparatively large, $b = 200$. At small values of the parameter b (Re), one observes large oscillations of positive and negative values of the velocity, that is, of the functions $y_n(x)$. Corresponding to these values are regions of inflow ($y_n > 0$) and outflow ($y_n > 0$). An increase in the parameter b (Re) reduces the amplitude of the oscillations and causes a reduction in back flows. At high values of $b \sim 10^2$–10^3 one observes pronounced forms corresponding to the flow of a weakly viscous fluid in a convergent channel. The deviation from the rectangular flow profile of an inviscid (ideal) fluid, in terms of a suitable metric, tends to zero for $0 < x < 1$; near $x = 0, 1$ ($\theta = \mp\beta$) one has typical boundary-layer phenomena.

Note that, according to the last relation of (5.1), the pressure profile p and flow velocity profile y are linearly related [14, 15], the principal component being $p_n^0 = \lambda_n/(a^2 b)$. The relative variations of the function $p = p_n(x) = p_n^0(1 + \delta_n(x))$ are very small and are estimated by the quantities $\delta_n = -a^2 y_n/\lambda_n$. Computations indicate (see Table 1) that the principal values of the pressure and estimates of the extremal values of δ_n^* with respect to x also have this property. Thus, the essential (qualitative) restructuring of the flow is caused by small variations in pressure, which decrease without limit as the parameter b, i.e. Re, is increased, where $p_n \to -a^{-2}$ as $b \to \infty$.

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